Experimental verification of conjectures on large classes of numerical semigroups: an overview

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Abstract: Let $\mathbb{N}$ denote the set of nonnegative integers. A numerical semigroup is a cofinite submonoid of $\mathbb{N}$ under addition. Throughout this abstract, S denotes an arbitrary numerical semigroup other than $\mathbb{N}$.

A numerical semigroup has several combinatorial invariants attached. Let $G(S) = \mathbb{N} \setminus S$. An element of $G(S)$ is called a *gap* of S and the cardinality of $G(S)$ is the *genus* of S. The maximum gap of S is called the *Frobenius number* of S and is denoted by $F(S)$. We convention that the Frobenius number of the only numerical semigroup with no gaps, $\mathbb{N}$, is 0. The least positive integer in S is called the *multiplicity* and is denoted $m(S)$. Let $S^* = S \setminus \{0\}$. It is well-known that $P(S) = S \setminus (S^* + S^*)$ is finite and is the unique minimal set of generators of S. Its elements are named the *primitives* of S, and the *maximum primitive* (maxprim, for short) plays a special role in our talk. The number of primitives of S, $|P(S)|$, is called the *embedding dimension* of S.

Next we refer to three partitions of the class of all numerical semigroups that are considered in the literature. Denote by G_n, F_n and A_n, respectively, the set of numerical semigroups with genus n, Frobenius number n and maximum primitive n. Let $g_n = |G_n|$, and, similarly, $f_n = |F_n|$, $a_n = |A_n|$. Associated to the partition $\{G_0, G_1, G_2, \ldots\}$ is the numerical sequence (g_n). The numerical sequences (f_n) and (a_n) are obtained analogously from the other partitions considered. The study of any of these sequences is generally referred to as *counting* by the name of the invariant that originates the sequence.

Wilf asked for the asymptotic behavior of the sequence (f_n) and Backelin answered Wilf's question. Brás-Amorós conjectured that (g_n) has an asymptotic behavior similar to the Fibonacci sequence, and Zhai proved this fact. The sequences (f_n) and (a_n) are Möbius transforms of one another, as proved in joint work with Kumar and Marion.

All the above numerical sequences exhibit exponential growth. Some effort has been made on computing the initial values of the sequences. The biggest exact values currently known are $a_{119} \simeq a_{120} \simeq f_{119} \simeq f_{120} \simeq 1.46 \times 10^{18}$ and $g_{77} \simeq 4.7 \times 10^{16}$.

Regarding counting by genus, various authors have proposed new algorithms and optimizations. Implementations in C or C++ are available. The record is due to Brás-Amorós. Regarding the counting by maximum primitive (and thus by Frobenius number), the number a_{120} was achieved using GAP code implementing algorithms obtained recently in joint work with N. Kumar.

The above numbers have been obtained by exploring a tree whose set of nodes contains all numerical semigroups. The classical tree is usually used for genus counting; a distinct tree of monoids was used for maxprim counting.

The process of counting is easily adapted to verify properties: one verifies the property at each explored node. In this case, one can incorporate additional theoretical results, some of which yield relatively efficient computations.

Recall that we are using S to denote an arbitrary numerical semigroup different from $\mathbb{N}$. The *conductor* of S is $c = c(S) = F(S) + 1$. We denote by $\ell = \ell(S)$ the number of left elements in S to the left of (i.e., smaller than) the Frobenius number.

The *Wilf number* of S is: $W = W(S) = \ell \times |P| - c$.

In equivalent terms, Wilf (1978) posed the following question: Does every numerical semigroup have a positive Wilf number?

We say that a numerical semigroup is *Wilf* if its Wilf number is nonnegative.

Wilf's conjecture: Every numerical semigroup is Wilf.

Our interest lies in verifying the nonexistence of counterexamples to Wilf's conjecture for numerical semigroups with as large as possible genus, Frobenius number and maxprim, that is, in showing that, for as large values of n as the computations permit, all numerical semigroups in G_n, F_n and A_n are Wilf.

Next, we refer to theoretical results that lead to efficient computations.

Theorem Numerical semigroups satisfying any of the following conditions are Wilf:

- $m \leq 19$ (Bruns *et al.* / Kliem and Stump);
- $\ell \leq 12$ (Eliahou and Marín Aragón);
- $3m \leq c$ (Eliahou)
  - these are the vast majority under genus or Frobenius counting (Zhai; Backelin/Singhal);
- $3|P| \geq m$ (Eliahou)
  - these are the vast majority under genus counting (Kaplan and Singhal);
- $|S \cap \{m, \ldots, 2m\}| \geq \sqrt{2m}$ (Delgado, Kumar and Marion)
  - these are the vast majority under maxprim counting.

With S. Eliahou and J. Fromentin, we obtained the most relevant result in this direction published up to now: there is no counter-example to Wilf's conjecture among the 6.78×10^{21} (estimated value) numerical semigroups with genus not greater than 100.

As a consequence, and also the fact that semigroups such that $\ell \leq 12$ are Wilf, one concludes that all the $273549070152290958 \simeq 2.7 \times 10^{17}$ numerical semigroups with Frobenius number up to 111 are Wilf.

With N. Kumar we showed, using GAP, that all the $5669829315454645 \simeq 5.6 \times 10^{15}$ numerical semigroups with maxprim up to 100 are Wilf.

Besides overviewing work regarding the experimental verification of Wilf's conjecture in the classes G_n, F_n and A_n for (not so) small values of n, this talk should illustrate a close connection between counting numerical semigroups and Wilf's conjecture – two questions posed by Wilf.

Some of the results referred to are part of a recent joint work with Neeraj Kumar. It is publicly available as a preprint. The same preprint contains references to most of the other results referred to in this abstract.

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